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Generating Asymptotics for factorially divergent sequences

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Abstract. The algebraic properties of formal power series with factorial growth which admit a certain well-behaved asymptotic expansion are discussed. These series form a subring of $\mathbb{R}[[x]]$ which is closed under composition. An 'asymptotic derivation' is defined which maps a power series to its asymptotic expansion. Leibniz and chain rules for this derivation are deduced. With these rules asymptotic expansions of implicitly defined power series can be obtained. The full asymptotic expansions of the number of connected chord diagrams and the number of simple permutations are given as examples.

Résumé. Nous discuterons des propriétés algébriques des séries entières formelles à croissance factorielle dont le développement asymptotique se comporte "bien". Ces séries forment un sous anneau de $\mathbb{R}[[x]]$, fermé sous composition. On définit une application de "dérivation asymptotique" qui associe à une série entière son développement asymptotique. On en déduit la formule de Leibniz et la dérivée de fonctions composées pour cette dérivation. Grâce à ces règles, les développements asymptotiques de séries entières implicitement définies peuvent être obtenus. Nous prendrons pour exemple les développements asymptotiques complets du nombre de diagrammes de cordes connexes et du nombre de permutations simples.

Keywords: asymptotic expansions, formal power series, chord diagrams, simple permutations

1 Introduction

This article is concerned with sequences a_n which admit an asymptotic expansion of the form,

$$a_n \sim \alpha^{n+\beta}\Gamma(n+\beta)\left(d_0 + \frac{d_1}{(n+\beta-1)} + \frac{d_2}{(n+\beta-1)(n+\beta-2)} + \dots\right),$$

for some $\alpha, \beta \in \mathbb{R}_{>0}$ and $d_k \in \mathbb{R}$. Sequences of this type appear in many enumeration problems, which deal with coefficients of factorial growth. For instance, generating functions of subclasses of permutations and graphs of fixed valence show this behaviour [1,

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7]. Furthermore, there are countless examples where *perturbative expansions* of physical quantities, for example in quantum mechanics, quantum field theory, statistical physics, matrix models and fluid dynamics, admit asymptotic expansions of this kind [4, 15, 20, 2, 18]. In this article these sequences will be interpreted as the coefficients of a formal power series. It is well known that formal power series $a \in \mathbb{R}[[x]]$ whose coefficients are bounded by some factorial growth $|a_n| \leq C\alpha^n \Gamma(n+\beta)$ are closed under addition, multiplication, inversion and composition. This article is an attempt to convince the reader that even more structure can be found if we restrict ourselves to formal power series which admit a well-behaved asymptotic expansion as above. This approach is inspired by the work of Bender [6] in which the asymptotic behaviour of the composition of a mildly growing power series with a rapidly growing power series is analyzed. These structures bear many resemblances to the theory of resurgence established by Jean Ecalle [16]. Resurgence assigns a special role to power series which diverge factorially, as they offer themselves to be Borel transformed. By Borel transformation, resurgence can be used to assign a unique function to a factorially divergent series, which could be interpreted as the generating function of the series. In fact, the presented formalism can be seen as a toy model of resurgence, which is unable to fully reconstruct functions from asymptotic expansions, but does not rely on Borel transform and offers itself for combinatorial applications. For an illuminating account on resurgence theory, we refer to David Sauzin's review [21]. In this extended abstract, we refer to [10] for the proofs.

1.1 Statement of results

Power series with well-behaved asymptotic expansions, as in the example above, form a subring of $\mathbb{R}[[x]]$, which will be denoted as $\mathbb{R}[[x]]^\alpha_\beta$. A linear map, $\mathcal{A}^\alpha_\beta : \mathbb{R}[[x]]^\alpha_\beta \to \mathbb{R}[[x]]$, can be defined which *maps a power series to the asymptotic expansion of its coefficients*. This map turns out to be a *derivation*, that means it fulfills a *Leibniz rule*

$$\text{with } f, g \in \mathbb{R}[[x]]^\alpha_\beta \quad (\mathcal{A}^\alpha_\beta(f \cdot g))(x) = f(x)(\mathcal{A}^\alpha_\beta g)(x) + (\mathcal{A}^\alpha_\beta f)(x) g(x)$$

$$\text{and a } \textit{chain rule,} \qquad (\mathcal{A}^\alpha_\beta(f \circ g))(x) = f'(g(x))(\mathcal{A}^\alpha_\beta g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}^\alpha_\beta f)(g(x)),$$

where $(f \cdot g)(x) = f(x)g(x)$ and $(f \circ g)(x) = f(g(x))$. Note that the chain rule involves a peculiar correction term if f has a non-trivial asymptotic expansion. The formalism can be applied to calculate the asymptotic expansions of implicitly defined power series. This procedure is similar to the extraction of the ordinary derivative of an implicitly defined function using the implicit function theorem. In Section 5, we demonstrate the apparatus by stating the full asymptotic expansions for the number of *connected chord diagrams* and for the number of *simple permutations*.

1.2 Notation

We will denote a (formal) power series $f \in \mathbb{R}[[x]]$ in the usual 'functional' notation $f(x) = \sum_{n=0}^{\infty} f_n x^n$. The coefficients of a power series f will be expressed by the same symbol with the index attached as a subscript f_n or with the coefficient extraction operator $[x^n]f(x) = f_n$. Ordinary (formal) derivatives are expressed as $f'(x) = \sum_{n=0}^{\infty} n f_n x^{n-1}$. The ring of power series which represent expansions of functions which are analytic at the origin, or equivalently power series with non-vanishing radius of convergence, will be denoted as $\mathbb{R}\{x\}$. We will make use of the $\mathcal{O}$-notation: $\mathcal{O}(a_n)$ denotes the set of all sequences b_n such that $\limsup_{n\to\infty} |\frac{b_n}{a_n}| < \infty$ and $o(a_n)$ all sequences b_n such that $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$. Equations of the form $a_n = b_n + \mathcal{O}(c_n)$ are to be interpreted as statements $a_n - b_n \in \mathcal{O}(c_n)$ as usual. See [5] for an introduction to this notation.

2 Prerequisites

We start by establishing a suitable notion of a power series with a well-behaved asymptotic expansion.

Definition 2.1. For given $\alpha, \beta \in \mathbb{R}_{>0}$ let $\mathbb{R}[[x]]^{\alpha}_{\beta}$ be the subset of $\mathbb{R}[[x]]$, such that $f \in \mathbb{R}[[x]]^{\alpha}_{\beta}$ if and only if there exists a sequence of real numbers $(c^f_k)_{k\in\mathbb{N}_0}$, which fulfills

$$f_n = \sum_{k=0}^{R-1} c^f_k \alpha^{n+\beta-k}\Gamma(n+\beta-k) + \mathcal{O}\left(\alpha^n \Gamma(n+\beta-R)\right) \qquad \forall R \in \mathbb{N}_0. \tag{2.1}$$

Corollary 2.1. *$\mathbb{R}[[x]]^{\alpha}_{\beta}$ is a linear subspace of $\mathbb{R}[[x]]$.*

Remark 2.1. The expression in (2.1) represents an asymptotic expansion or Poincaré expansion with the asymptotic scale $\alpha^n\Gamma(n+\beta)$ [14, Ch. 1.5].

Remark 2.2. The subspace $\mathbb{R}[[x]]^{\alpha}_{\beta}$ includes all (real) power series whose coefficients only grow exponentially: $\mathbb{R}\{x\} \subset \mathbb{R}[[x]]^{\alpha}_{\beta}$.

The central idea in this article is to *interpret the coefficients c^f_k in the asymptotic expansion as another power series*. Therefore, we define a linear map $\mathcal{A}^{\alpha}_{\beta}$ on $\mathbb{R}[[x]]^{\alpha}_{\beta}$, which maps a formal power series to its asymptotic expansion:

Definition 2.2. Let $\mathcal{A}^{\alpha}_{\beta} : \mathbb{R}[[x]]^{\alpha}_{\beta} \to \mathbb{R}[[x]]$ be the map that associates a power series $\mathcal{A}^{\alpha}_{\beta} f \in \mathbb{R}[[x]]$ to every power series $f \in \mathbb{R}[[x]]^{\alpha}_{\beta}$ such that

$$f_n = \sum_{k=0}^{R-1} \alpha^{n+\beta-k}\Gamma(n+\beta-k)[x^k](\mathcal{A}^{\alpha}_{\beta} f)(x) + \mathcal{O}\left(\alpha^n\Gamma(n+\beta-R)\right). \tag{2.2}$$

Example 2.1. The power series $f \in \mathbb{R}[[x]]$ associated to the sequence $f_n = n!$ clearly fulfills the requirements of Definition 2.1 with $\alpha = 1$ and $\beta = 1$. Therefore, $f \in \mathbb{R}[[x]]^1_1$ and $(\mathcal{A}^1_1 f)(x) = 1$.

3 A derivation for asymptotics

Proposition 3.1. *$\mathbb{R}[[x]]^\alpha_\beta$ forms a subring of $\mathbb{R}[[x]]$: If $f, g \in \mathbb{R}[[x]]^\alpha_\beta$, then $f \cdot g \in \mathbb{R}[[x]]^\alpha_\beta$. Moreover, $\mathcal{A}^\alpha_\beta$ is a derivation:*

$$(\mathcal{A}^\alpha_\beta(f \cdot g))(x) = f(x)(\mathcal{A}^\alpha_\beta g)(x) + g(x)(\mathcal{A}^\alpha_\beta f)(x). \tag{3.1}$$

Sketch of proof. Set $h(x) = f(x)g(x)$ and $R \in \mathbb{N}_0$ with $n > 2R$. We can rewrite the usual Cauchy product formula for power series as

$$h_n = \sum_{m=0}^{n} f_{n-m} g_m = \sum_{m=0}^{R-1} f_{n-m} g_m + \sum_{m=0}^{R-1} f_m g_{n-m} + \sum_{m=R}^{n-R} f_m g_{n-m}. \tag{3.2}$$

Definition 2.2 guarantees that the first two sums have an asymptotic expansion as in Definition 2.1 for large n. The sum of both constitutes an asymptotic expansion of h_n. It is sufficient to verify this claim for the first sum, where we substitute the asymptotic expansion from (2.2) of f_{n-m} up to order $R - m$:

$$\sum_{m=0}^{R-1} f_{n-m} g_m = \sum_{k=0}^{R-1} \alpha^{n+\beta-k} \Gamma(n + \beta - k) \sum_{m=0}^{k} g_m c_{k-m} + \mathcal{O}\left(\alpha^n \Gamma(n + \beta - R)\right). \tag{3.3}$$

The inner sum $\sum_{m=0}^{k} g_m c_{k-m}$ is the k-th coefficient of the product $g(x)(\mathcal{A}^\alpha_\beta f)(x)$.

It remains to be shown that the last sum in (3.2) is negligible. It can be done using basic properties of the Γ-function. □

4 Composition of power series in $\mathbb{R}[[x]]^\alpha_\beta$

The following theorem is a straightforward generalization of Bender's Theorem 1 in [6] to the multivariate case $f \in \mathbb{R}\{y_1, \ldots, y_L\}$.

Theorem 4.1. *If $g^1, \ldots, g^L \in \mathbb{R}[[x]]^\alpha_\beta$ with $g^l_0 = 0$ for $l \in \{1, \ldots, L\}$ and $f \in \mathbb{R}\{y_1, \ldots, y_L\}$, a function in L variables, which is analytic at the origin, then with $h(x) = f(g^1(x), \ldots, g^L(x))$ the power series h is in $\mathbb{R}[[x]]^\alpha_\beta$ and*

$$(\mathcal{A}^\alpha_\beta h)(x) = (\mathcal{A}^\alpha_\beta(f(g^1, \ldots, g^L)))(x) = \sum_{l=1}^{L} \frac{\partial f}{\partial g^l}(g^1, \ldots, g^L)(\mathcal{A}^\alpha_\beta g^l)(x). \tag{4.1}$$

It is obvious that a general chain rule cannot be as simple as the ordinary chain rule for differentiation. For general f, g: $(\mathcal{A}^\alpha_\beta(f \circ g))(x) \neq f'(g(x))(\mathcal{A}^\alpha_\beta g)(x)$. Otherwise, the requirement that the generating function $g(x) = x$ has a trivial asymptotic expansion, $(\mathcal{A}^\alpha_\beta g)(x) = 0$, would imply that all $f \in \mathbb{R}[[x]]^\alpha_\beta$ have trivial asymptotic expansions. We can fix that by including a correction term into the chain rule:

Theorem 4.2. *If $f, g \in \mathbb{R}[[x]]^\alpha_\beta$ with $g_0 = 0, g_1 = 1$, then $f \circ g, g^{-1} \in \mathbb{R}[[x]]^\alpha_\beta$ and*

$$(\mathcal{A}^\alpha_\beta(f \circ g))(x) = f'(g(x))(\mathcal{A}^\alpha_\beta g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} (\mathcal{A}^\alpha_\beta f)(g(x)), \tag{4.2}$$

$$(\mathcal{A}^\alpha_\beta g^{-1})(x) = -g^{-1\prime}(x) \left(\frac{x}{g^{-1}(x)}\right)^\beta e^{\frac{g^{-1}(x)-x}{\alpha x g^{-1}(x)}} (\mathcal{A}^\alpha_\beta g)(g^{-1}(x)). \tag{4.3}$$

Remark 4.1. We refer to [10] for the proof. It is technical and exploits the Lagrange inversion formula to express $f \circ g^{-1}$ as,

$$[x^n] f(g^{-1}(x)) = \sum_{m=0}^{n} \binom{n+\beta+1}{m} [x^{n-m}] B(x) A(x)^m \qquad \forall n \in \mathbb{N}_0,$$

with $A(x) := \frac{\frac{x}{g(x)} - 1}{x}$ and $B(x) := f(x) g'(x) \left(\frac{g(x)}{x}\right)^\beta$. The left hand side gives a good approximation for large n asymptotics if we sum over the first $C \log n$ terms, where C depends on f and g. This can be used to obtain the complete asymptotics of $[x^n] f \circ g^{-1}(x)$. The statement of the theorem follows immediately from this.

Remark 4.2. Bender and Richmond [8] established that $[x^n](1 + g(x))^{\gamma n + \delta} = n \gamma e^{\frac{\gamma g_1}{\alpha}} g_n + \mathcal{O}(g_n)$ if $g \in \mathbb{R}[[x]]^\alpha_\beta$. Using Lagrange inversion the first coefficient in the expansion of the compositional inverse in (4.3) can be obtained from this. Therefore, Theorem 4.2 can be seen as a generalization of Bender and Richmond's result. In the same article Bender and Richmond proved a similar theorem as Theorem 4.2 for power series f which grow more rapidly than the factorial such that $n f_{n-1} \in o(f_n)$. Theorem 4.2 establishes a link to the excluded case $n f_{n-1} = \mathcal{O}(f_n)$.

Remark 4.3. The chain rule

$$(\mathcal{A}^\alpha_\beta(f \circ g))(x) = (f' \circ g)(x)(\mathcal{A}^\alpha_\beta g)(x) + \left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}} ((\mathcal{A}^\alpha_\beta f) \circ g)(x) \tag{4.4}$$

exposes a peculiar algebraic structure. It would be useful to have a combinatorial interpretation of the $\left(\frac{x}{g(x)}\right)^\beta e^{\frac{g(x)-x}{\alpha x g(x)}}$ term.

5 Applications

5.1 Chord diagrams

A chord diagram with n-chords is a circle with $2n$ points, which are labeled with integers $1, \ldots, 2n$ and connected in disjoint pairs by n-chords. There are $(2n-1)!!$ of such diagrams.

5.1.1 Connected diagrams

A chord diagram is *connected* if no set of chords can be separated from the remaining chords by a line which does not cross any chords. Let $I(x) = \sum_{n=0}(2n-1)!!x^n$ and $C(x) = \sum_{n=0} C_n x^n$, where C_n is the number of connected chord diagrams with n chords. $I(x)$ and $C(x)$ are related by,

$$I(x) = 1 + C(xI(x)^2). \tag{5.1}$$

See for instance [17] for a proof. This functional equation can be solved for the coefficients of $C(x)$ by basic iterative methods. The first few terms are,

$$C(x) = x + x^2 + 4x^3 + 27x^4 + 248x^5 + \cdots \tag{5.2}$$

This sequence is entry A000699 in Neil Sloane's integer sequence on-line encyclopedia [22].

Because $(2n-1)!! = \frac{2^{n+\frac{1}{2}}}{\sqrt{2\pi}}\Gamma(n+\frac{1}{2})$, we have $I, C \in \mathbb{R}[[x]]^2_{\frac{1}{2}}$ and $(\mathcal{A}^2_{\frac{1}{2}} I)(x) = \frac{1}{\sqrt{2\pi}}$. Application of Theorem 4.2 gives,

$$(\mathcal{A}^2_{\frac{1}{2}} I)(x) = 2xI(x)C'(xI(x)^2)(\mathcal{A}^2_{\frac{1}{2}} I)(x) + \left(\frac{x}{xI(x)^2}\right)^{\frac{1}{2}} e^{\frac{xI(x)^2 - x}{2x^2 I(x)^2}} (\mathcal{A}^2_{\frac{1}{2}} C)(xI(x)^2) \tag{5.3}$$

Using (5.1), we obtain

$$(\mathcal{A}^2_{\frac{1}{2}} C)(x) = \frac{1 + C(x) - 2xC'(x)}{\sqrt{2\pi}} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}. \tag{5.4}$$

We can still simplify this using the differential equation $C'(x) = \frac{C(x)(1+C(x))-x}{2xC(x)}$ which follows from the linear differential equation $2x^2 I'(x) + xI(x) + 1 = I(x)$ and (5.1),

$$(\mathcal{A}^2_{\frac{1}{2}} C)(x) = \frac{1}{\sqrt{2\pi}} \frac{x}{C(x)} e^{-\frac{1}{2x}(2C(x)+C(x)^2)}. \tag{5.5}$$

This is the full asymptotic expansion of C_n. The first few terms are,

$$(\mathcal{A}^2_{\frac{1}{2}} C)(x) = \frac{e^{-1}}{\sqrt{2\pi}} \left(1 - \frac{5}{2}x - \frac{43}{8}x^2 - \frac{579}{16}x^3 - \frac{44477}{128}x^4 - \frac{5326191}{1280}x^5 + \cdots\right). \tag{5.6}$$

Expressed in the traditional way using (2.2) this becomes

$$C_n \sim \sum_{k\geq 0} 2^{n+\frac{1}{2}-k}\Gamma(n+\frac{1}{2}-k)[x^k](\mathcal{A}^2_{\frac{1}{2}}C)(x) = \sqrt{2\pi}\sum_{k\geq 0}(2(n-k)-1)!![x^k](\mathcal{A}^2_{\frac{1}{2}}C)(x) \tag{5.7}$$

$$\begin{aligned} &= e^{-1}\Bigg((2n-1)!! - \frac{5}{2}(2n-3)!! - \frac{43}{8}(2n-5)!! \\ &- \frac{579}{16}(2n-7)!! - \frac{44477}{128}(2n-9)!! - \frac{5326191}{1280}(2n-11)!! + \cdots\Bigg) \end{aligned} \tag{5.8}$$

The first term, e^{-1}, in this expansion has been computed by Kleitman [19], Stein and Everett [23] and Bender and Richmond [8] each using different methods. With the presented method an arbitrary number of coefficients can be computed.

The probability of a random chord diagram with n chords to be connected is therefore $e^{-1}(1-\frac{5}{4n}) + \mathcal{O}(\frac{1}{n^2})$.

5.1.2 Monolithic diagrams

A chord diagram is called monolithic if it consists only of a connected component and of isolated chords which do not 'contain' each other [17]. That means with (a, b) and (c, d) the labels of such edges we do not allow $a < c < d < b$ or $c < a < b < d$. Let $M(x) = \sum_{n=0} M_n x^n$ be the generating function of monolithic chord diagrams. Following [17], $M(x)$ fulfills

$$M(x) = C\left(\frac{x}{(1-x)^2}\right). \tag{5.9}$$

Using the $\mathcal{A}^2_{\frac{1}{2}}$ derivative on both sides of this equation together with the result for $(\mathcal{A}^2_{\frac{1}{2}}C)(x)$ in (5.5) gives

$$\begin{aligned} (\mathcal{A}^2_{\frac{1}{2}}M)(x) &= \frac{1}{\sqrt{2\pi}}\frac{1}{(1-x)}\frac{x}{M}e^{1-\frac{x}{2}-\frac{(1-x)^2}{2x}(2M(x)+M(x)^2)} \\ &= \frac{1}{\sqrt{2\pi}}\left(1-4x-6x^2-\frac{154}{3}x^3-\frac{1610}{3}x^4-\frac{34588}{5}x^5+\cdots\right). \end{aligned} \tag{5.10}$$

The probability of a random chord diagram to be monolithic is therefore $1-\frac{4}{2n-1}+\mathcal{O}(\frac{1}{n^2}) = 1-\frac{2}{n}+\mathcal{O}(\frac{1}{n^2})$.

5.2 Simple permutations

A permutation is called simple if it does not map a non-trivial interval to another interval. Expressed formally, the permutation $\pi \in S_n^{\text{simple}} \subset S_n$ if and only if $\pi([i, j]) \neq [k, l]$

for all $i,j,k,l \in [1,n]$ with $2 \leq |[i,j]| \leq n-1$. See Albert et al. [1] for more detailed exposition of simple permutations or [12] for a more recent survey. Set $S(x) = \sum_{n=4}^\infty |S_n^{\text{simple}}| x^n$ and $F(x) = \sum_{n=1}^\infty n! x^n$. Following [1], $S(x)$ and $F(x)$ fulfill the functional equation

$$\frac{F(x) - F(x)^2}{1+F(x)} = x + S(F(x)). \tag{5.11}$$

This can be solved iteratively for the coefficients of $S(x)$:

$$S(x) = 2x^4 + 6x^5 + 46x^6 + 338x^7 + 2926x^8 + \cdots \tag{5.12}$$

In Neil Sloane's online encyclopedia this sequence is entry `A111111` [22] with a slightly different convention: `A111111` $= x + 2x^2 + S(x)$.

As $F(x) \in \mathbb{R}[[x]]_1^1$ and $(\mathcal{A}_1^1 F) = 1$, we obtain the full asymptotic expansion of $S(x)$ after application of Theorem 4.2,

$$(\mathcal{A}_1^1 F)(x) = F'(x) \frac{x}{F(x)} e^{\frac{F(x)-x}{xF(x)}} (\mathcal{A}_1^1 S)(F(x)). \tag{5.13}$$

Using the functional equation (5.11) or the compositional inverse of $F(x)$, $F^{-1}(F(x)) = x$ as well as the differential equation $x^2 F'(x) + (x-1)F(x) + x = 0$, it is straightforward to solve this for $(\mathcal{A}_1^1 S)(x)$,

$$(\mathcal{A}_1^1 S)(x) = {F^{-1}}'(x) \frac{x}{F^{-1}(x)} e^{\frac{F^{-1}(x)-x}{xF^{-1}(x)}} = \frac{xF^{-1}(x)}{x - (1+x)F^{-1}(x)} e^{\frac{F^{-1}(x)-x}{xF^{-1}(x)}} \tag{5.14}$$

$$= \frac{1}{1+x} \frac{1 - x - (1+x)\frac{S(x)}{x}}{1 + (1+x)\frac{S(x)}{x^2}} e^{-\frac{2+(1+x)\frac{S(x)}{x^2}}{1-x-(1+x)\frac{S(x)}{x}}}. \tag{5.15}$$

Note that (5.11) implies that $\frac{x-x^2}{1+x} = F^{-1}(x) + S(x)$, which means $F^{-1}(x)$ and $S(x)$ differ only by an asymptotically negligible quantity, (i.e. some $p \in \ker \mathcal{A}_1^1$) and $(\mathcal{A}_1^1 F^{-1})(x) = -(\mathcal{A}_1^1 S)(x)$.

The terms of $(\mathcal{A}_1^1 S)(x)$ can easily be computed iteratively:

$$(\mathcal{A}_1^1 S)(x) = e^{-2} \left(1 - 4x + 2x^2 - \frac{40}{3}x^3 - \frac{182}{3}x^4 - \frac{7624}{15}x^5 + \cdots \right). \tag{5.16}$$

By (2.2), this is an expression for the asymptotics of the number of simple permutations:

$$|S_n^{\text{simple}}| \sim e^{-2} n! \left(1 - 4\frac{1}{n} + 2\frac{1}{n(n-1)} - \frac{40}{3}\frac{1}{n(n-1)(n-2)} - \frac{182}{3}\frac{1}{n(n-1)(n-2)(n-3)} - \frac{7624}{15}\frac{1}{n(n-1)(n-2)(n-3)(n-4)} + \cdots \right). \tag{5.17}$$

Albert et al. [1] calculated the first three terms of this expansion. With the presented methods the calculation of the asymptotic expansion $(\mathcal{A}_1^1 S)(x)$ up to order n is as easy as calculating the expansion of $S(x)$ or $F^{-1}(x)$ up to order $n+2$.

Remark 5.1. The examples above are chosen to demonstrate that given a (functional) equation which relates two power series in $\mathbb{R}[[x]]^\alpha_\beta$, it is an easy task to calculate the full asymptotic expansion of one of the power series from the asymptotic expansion of the other power series.

Applications include functional equations for 'irreducible combinatorial objects'. The two examples fall into this category. Irreducible combinatorial objects were studied in general by Beissinger [3].

Remark 5.2. Dyson-Schwinger equations in quantum field theory can be stated as functional equations of this form [13, 9]. In [11] we elaborated on this idea in the scope of zero-dimensional quantum field theory and the associated graph counting problems.

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